What Is Normal Mode Decomposition?

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1 Introduction

Roughly speaking, Normal Mode Decomposition (NMD) is merely a physical application of Singular Value Decomposition (SVD).

In January 2024, a paper [1] hypothesized that neurons can do NMD. To me, it is better to say neurons can do SVD. Anyway, whatever they like.

NMD is not mystery, let me clarify it.

2 Balls and Springs

This section is based on a lecture note of David Morin.

2.1 1 Ball

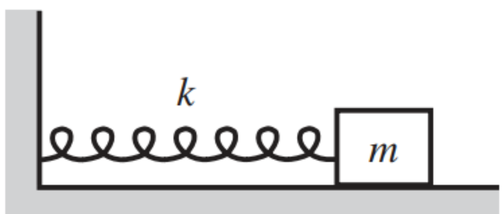

Figure 1: One Ball[2]

Physicists call things like Figure 1 simple harmonics, which has important applications in both classical quantum mechanics.

The ODE of Figure 1 is:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0 \tag{1}$$

, where $\omega := \sqrt{\frac{k}{m}}$.

Its solution is:

$$x(t) = a_1 cos(\omega t) + a_2 sin(\omega t) \tag{2}$$

, where a_1 and a_2 are determined by initial conditions (i.e. $x(0)$ and $v(0)$).

Another insightful way is to implement $\frac{\mathrm{d}x}{\mathrm{d}t}$ as a new variable. Define

$$X := \begin{bmatrix} x \\ \frac{\mathrm{d}x}{\mathrm{d}t} \end{bmatrix} \tag{3}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} X = AX \tag{4}$$

, where

$$A := \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \tag{5}$$

The fixed point of this system is $X = [0; 0]$ and it is unstable (stable fixed point means returning to this point after small perturbation and here if you perturb the ball a little, it will move forever. Of course in reality this is impossible, that is because there is a friction force $f \propto \dot{x}$ making this fixed point stable).

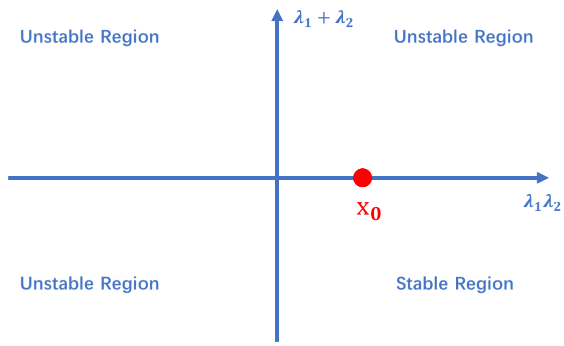

Figure 2: Determine the stability of the fixed point

This approach is called dynamical system.

Here comes a question: does this system have one dimension or two dimensions? Similar questions also are raised in the subsequent sub-sections. I will clarify it in subsection 3.1.

2.2 2 Balls

Now we shift to two balls. A little complicated. We introduce a symmetry (The stiffness coefficients of the 1st and 3rd spring are the same) to simplify this problem.

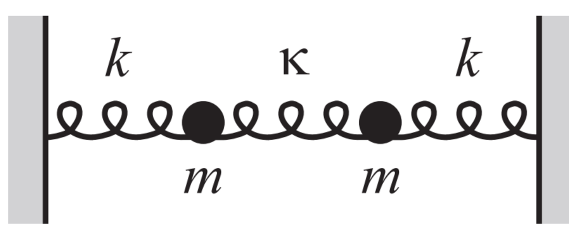

Figure 3: Two Balls

The ODEs is:

$$\begin{cases} m\ddot{x}_1 = -kx_1 - \kappa(x_1 - x_2) \\ m\ddot{x}_2 = -kx_2 - \kappa(x_2 - x_1) \end{cases} \tag{6}$$

You can guess that (x_1+x_2) and (x_1-x_2) will take the form of Equation 1, i.e. oscillate like simple harmonics.

$$\begin{cases} m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x_1 + x_2) = -k(x_1 + x_2) \\ m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x_1 - x_2) = -(k + 2\kappa)(x_1 - x_2) \end{cases} \tag{7}$$

(x_1+x_2) and (x_1-x_2) are called normal mode or normal coordinates. Check subsection 3.2 for the strict definition of normal modes. Check subsection 3.2 for the number of normal modes in this situation.

Below, I will introduce a general way to find this kind of harmonics.

Rewrite Equation 6 in matrix form:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}X = AX \tag{8}$$

, where

$$X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9}$$

and

$$A = \frac{1}{m}\begin{bmatrix} -(k+\kappa) & \kappa \\ \kappa & -(k+\kappa) \end{bmatrix} \tag{10}$$

Mathematicians tell us the solution of Equation 8 is in this form:

$$X = Ce^{\lambda t} \tag{11}$$

Plug in, we get:

$$\lambda^2 C = AC \tag{12}$$

Define $z := \lambda^2$, then

$$zC = AC \tag{13}$$

So, z is the eigen-value of A and C is the eigen-vector of A.

We can calculate the eigen-value and eigen-vector of A using standard pipeline. We will get:

$$z_1 = -\frac{k}{m},\ z_2 = -\frac{k+2\kappa}{m} \tag{14}$$

and

$$C_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{15}$$

Then

$$\lambda_1 = i\sqrt{\frac{k}{m}},\ \lambda_2 = -i\sqrt{\frac{k}{m}},\ \lambda_3 = i\sqrt{\frac{k+2\kappa}{m}},\ \lambda_4 = -i\sqrt{\frac{k+2\kappa}{m}} \tag{16}$$

Mathematicians also tell us the solution space of Equation 8 is:

$$span(\begin{bmatrix}1\\1\end{bmatrix}e^{\lambda_1 t}, \begin{bmatrix}1\\1\end{bmatrix}e^{\lambda_2 t}, \begin{bmatrix}1\\-1\end{bmatrix}e^{\lambda_3 t}, \begin{bmatrix}1\\-1\end{bmatrix}e^{\lambda_4 t}) \tag{17}$$

We can denote them as $V_1,\ V_2,\ V_3,\ V_4$ to make things more clear:

$$span(V_1, V_2, V_3, V_4) \tag{18}$$

The dim of $C_1 e^{\lambda_1 t}$ et al. 2 while the dim of the solution space is 4.

And we know that the space of Equation 17 equals to (check subsection 3.5 for the proof)

$$span(\begin{bmatrix}1\\1\end{bmatrix}cos(\omega_1 t), \begin{bmatrix}1\\1\end{bmatrix}sin(\omega_1 t), \begin{bmatrix}1\\-1\end{bmatrix}cos(\omega_2 t), \begin{bmatrix}1\\-1\end{bmatrix}sin(\omega_2 t)) \tag{19}$$

, where $\omega_1 := \sqrt{\frac{k}{m}}$, $\omega_2 := \sqrt{\frac{k+2\kappa}{m}}$

Also, we denote them as $\tilde{V}_1,\ \tilde{V}_2,\ \tilde{V}_3,\ \tilde{V}_4$ to make things more clear:

$$span(\tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{V}_4) \tag{20}$$

Finally we get the solution to Equation 6:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}cos(\omega_1 t) & sin(\omega_1 t) & cos(\omega_2 t) & sin(\omega_2 t)\\ cos(\omega_1 t) & sin(\omega_1 t) & -cos(\omega_2 t) & -sin(\omega_2 t)\end{bmatrix}\begin{bmatrix}a_1 & a_1\\a_2 & a_2\\a_3 & a_3\\a_4 & a_4\end{bmatrix} \tag{21}$$

, where $a_1,\ a_2,\ a_3,\ a_4$ are determined by the initial conditions (i.e. $x_1(0),\ v_1(0),\ x_2(0),\ v_2(0)$).

We can also write Equation 21 in a more clear way:

$$X = \begin{bmatrix}V_1 & V_2 & V_3 & V_4\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} \tag{22}$$

Check here for a real world experiment.

2.3 3 Balls

Now we consider 3 Balls.

The ODEs are

$$\begin{cases} m_1\ddot{x}_1 = -kx_1 - k(x_1 - x_2)\\ m_2\ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3)\\ m_3\ddot{x}_3 = -k(x_3 - x_2) - kx_3 \end{cases} \tag{23}$$

Rewrite:

$$\frac{d^2}{dt^2}X = AX \tag{24}$$

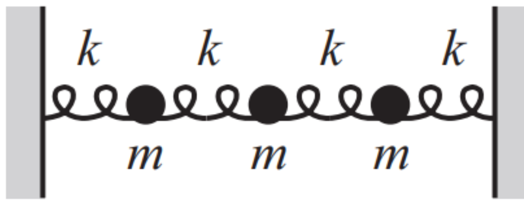

Figure 4: Three Balls

, where

$$A = \frac{1}{m}\begin{bmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -2k \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{25}$$

The eigen-values and eigen-vectors of A are:

$$z_1 = -2\frac{k}{m},\ z_2 = -(2+\sqrt{2})\frac{k}{m},\ z_3 = -(2-\sqrt{2})\frac{k}{m} \tag{26}$$

$$C_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix},\ C_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \tag{27}$$

The solution is:

$$X = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \tag{28}$$

, where V_i takes the form of $Ce^{\pm\sqrt{z}t}$ and a_i is determined by the initial conditions.

Check the 27s of this video for a real-world experiment.

2.4 N Balls

ODEs:

$$m\frac{d^2}{dt^2}\begin{bmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & & & & & \\ \ddots & k & -2k & k & & & \\ & & k & -2k & k & & \\ & & & k & -2k & k & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{bmatrix} \tag{29}$$

Rewrite:

$$\frac{d^2}{dt^2}X = AX \tag{30}$$

Theoretically, you can calculate the eigen-values and eigen-vectors of A like in Equation 26 and Equation 27 and then get the solution like in Equation 28.

2.5 Infinite Balls

Now let $N \to \infty$. Define $\xi(x,t)$ as the deviance of the ball in x position at time t. Then:

$$m\ddot{\xi}(x,t) = k\xi(x-\Delta x,t) - 2k\xi(x-\Delta x,t) + k\xi(x+\Delta x,t) \tag{31}$$

So,

$$(\frac{m}{\Delta x})\ddot{\xi}(x,t) = (k\Delta x)\frac{\frac{\xi(x+\Delta x,t)-\xi(x,t)}{\Delta x} - \frac{\xi(x,t)-\xi(x-\Delta x,t)}{\Delta x}}{\Delta x} \tag{32}$$

Denote $\rho := \frac{m}{\Delta x}$, $E := k\Delta x$ and $C := \frac{E}{\rho}$, then:

$$\frac{\partial^2}{\partial x^2}\xi(x,t) = C\frac{\partial^2}{\partial t^2}\xi(x,t) \tag{33}$$

Physicists call things like Equation 33 wave equations.

It is easy to understand the physical picture of $N \to \infty$ balls. In your mind, you can consider any longitudinal waves as infinite balls and springs.

3 Appendix

3.1 Clarify Dimensions

What is the dimensionality of a system of Ordinary Differential Equations? We can think this question from the point of physics and math.

In physics, for subsection 2.1, both $dim = 1$ and $dim = 2$ are right. That depends on the perspective you choose. Because, in physics, a definition is useful if and only if it brings measurable results. If you want to determine the position of the system, you need to measure 1 variable.If you want to determine the system (i.e. knowing how the system will evolve in the future), you need to measure 2 variables. This is similar to, if you want to determine the position of a rigid body, you need to measure 6 variables. If you want to determine the position and the speed of a rigid body, you need to measure 12 variables.

In math, when a mathematician talks about the dimensionality of a system of Ordinary Differential Equations, he means the dimensionality of the solution space. For subsection 2.1, the solution space is $S := \{a_1 cos(x) + a_2 sin(x) | a_1, a_2 \in \mathbb{R}^1\} = span(cos(x), sin(x))$, so the answer is 2.

3.2 The Strict Definition of Normal Modes

For discrete system, the strict definition of Normal Modes is: if

$$X = Cf(t) \tag{34}$$

where C is a constant vector, independent of time, then we call $\vec{X}$ a normal mode. Check here for the definition. Check here for a visualization via simulation experiments of a double pendulum.

For continuous system, like subsection 2.5, the definition is: if

$$\xi(x, t) = X(x)T(t) \tag{35}$$

then we call $\xi(x, t)$ a normal mode. Check here for animations.

3.3 Clarify the Number of Normal Modes

For subsection 2.2, the answer is 2.

Why is the dimensionality of the solution space 4, while the number of normal modes is only 2? Because, in Equation 19, $\tilde{V_1}$ and $\tilde{V_2}$ represent the same normal mode, while $\tilde{V_3}$ and $\tilde{V_4}$ correspond to another.

The physical picture is clear: in the real world, $sin(\omega t)$ and $cos(\omega t)$ represent the same oscillation with the only difference being a phase shift.

3.4 Orthogonality of Normal Coordinates

Eigenvectors like $[1; 1]$ in subsection 2.2 and $[1; 0; -1]$ in subsection 2.3 are called normal coordinates. They have the following properties:

- One normal coordinate corresponds to one normal mode.
- They are perpendicular to one another. Physically, this means the movement of a normal mode won't interfere with another.
- They form a basis of the solution space after multiplied by $e^{\pm\sqrt{z}t}$. Physically, this means any vibrations of the system can be linearly synthesized by normal modes.
- You could do normalization to make them to be an ortho-normal basis of the solution space.

By the way, orthogonality of normal coordinates can be used to give a wrong but insightful explanation of the motion of H atom and O atom in H_2O. Check here.

3.5 Linear Transformation Between Exp and Cos-Sin-Pair

We shall only prove the 1 dim situation. For multi-dims, the proof is similar. Suppose

$$x = a_1 e^{i\omega t} + a_2 e^{-i\omega t} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{bmatrix} \tag{36}$$

Plug in

$$\begin{bmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} cos(\omega t) \\ sin(\omega t) \end{bmatrix} \tag{37}$$

We get

$$x = \tilde{a}_1 cos(\omega t) + \tilde{a}_2 sin(\omega t) = \begin{bmatrix} \tilde{a}_1 & \tilde{a}_2 \end{bmatrix} \begin{bmatrix} cos(\omega t) \\ sin(\omega t) \end{bmatrix} \tag{38}$$

, where

$$\begin{bmatrix} \tilde{a}_1 & \tilde{a}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix} \tag{39}$$

So

$$span(e^{i\omega t}, e^{-i\omega t}) = span(cos(\omega t), sin(\omega t)) \tag{40}$$

3.6 Singular Value Decomposition

In this section, I will show you why NMD (or Eigen Value Decomposition, EMD) is a special kind of SVD.

SVD Theorem:

$$\forall A_{n\times m} \in \mathbb{R}^{n\times m}, A_{n\times m} = U_{n\times r} D_{r\times r} V^T_{r\times m} \tag{41}$$

, where r is the rank of A and

$$\begin{cases} D = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_r}), \lambda_i \text{ are eigen-vectors of } AA^T \text{ and } A^T A \\ U = [u_1, u_2, \cdots, u_r], u_i \text{ are eigen-vectors of } AA^T \\ V = [v_1, v_2, \cdots, v_r], v_i \text{ are eigen-vectors of } A^T A \\ U^T U = I,\ V^T V = I \end{cases} \tag{42}$$

Now you know why Eigen Value Decomposition is just a special kind of SVD.

3.7 Normal Mode Decomposition and Dynamic Mode Decomposition

To me, both NMD and DMD are indeed Singular Value Decomposition (SVD) in math form (Eigen Value Decomposition is a special kind of SVD). The distinction is merely the former one is usually used for transformations in low dimensions ($\sim 10^0$, in fact, usually 2) while the latter one is usually used for reducing high dimensions ($\sim 10^6$) to low dimensions ($\sim 10^2$).

Check this video at 10:25 to see the exact math form of DMD.

References

[1] Siavash Golkar, Jules Berman, David Lipshutz, Robert Mihai Haret, Tim Gollisch, and Dmitri B Chklovskii. Neuronal temporal filters as normal mode extractors. *Physical Review Research*, 6(1):013111, 2024.

[2] David Morin. *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.